

The University of Chicago

PROPERTIES OF SURFACES WHOSE OSCULATING
RULED SURFACES BELONG TO
LINEAR COMPLEXES

A DISSERTATION
SUBMITTED TO THE FACULTY
OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY
EDGAR D. MEACHAM

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I. INTRODUCTION

The fundamental equations and properties employed in the following investigation were developed by Professor Wilczynski and set forth by him in his first and second memoirs on the projective differential geometry of curved surfaces* and in his book on projective differential geometry.†

The author wishes here to express his thanks for the helpful advice so kindly given him by Professor Wilczynski, who suggested this subject and under whose direction this paper was prepared.

A surface S , referred to its asymptotic net, may be regarded as an integral surface of a completely integrable system of partial differential equations of the form:‡

$$(1) \quad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

the four independent solutions being regarded as the homogeneous coordinates of a point P_y of S .

The coefficients a' , b , f , g are seminvariants and they are connected by the integrability conditions

$$(2) \quad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0. \end{aligned}$$

The surface S is ruled if, and only if, $a'b = 0$.

* There are five of these memoirs which appeared in the *Transactions of the American Mathematical Society* (1907-10), hereinafter referred to as First memoir, Second memoir, etc.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Surfaces*, hereinafter referred to as W.

‡ First memoir, p. 223.

The functions

$$(3) \quad y, \quad y_u = z, \quad y_v = \rho, \quad y_{uv} = \sigma$$

are semicovariants, and the substitution of the four independent solutions of (1) in these expressions determines four points $P_y, P_z, P_\rho, P_\sigma$ which are the vertices of a non-degenerate semicovariant tetrahedron of reference T .*

At each point P_y of S there are two asymptotic tangents, which join P_y to P_z and P_ρ respectively. The ruled surface $R_1(u)$ generated by $P_y P_z$ as u remains constant and v assumes all its values is called the osculating ruled surface of the first kind.† The osculating ruled surface of the second kind, $R_2(v)$, is generated by $P_y P_\rho$ as v remains constant while u assumes all its values. The system of differential equations which characterizes R_1 is ‡

$$(4) \quad \begin{aligned} y_{vv} + g y + 2a'z &= 0 \\ z_{vv} - 4a'by_v + (g_u - 2a'f)y + (g + 2a'_u)z &= 0. \end{aligned}$$

The equations of R_2 are obtained from (4) by the interchanges

$$(5) \quad \begin{pmatrix} a' & f & z & u \\ b & g & \rho & v \end{pmatrix}.$$

The curves $u = \text{constant}$ and $v = \text{constant}$ are asymptotic on R_1 and R_2 , respectively, as well as on S .

The same quadric, Q_0 , osculates R_1 and R_2 at P_y and its equation referred to T is§

$$x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0.$$

In this paper we propose to study the properties of a non-ruled surface whose osculating ruled surfaces belong to linear complexes. An important special case, in which the asymptotic curves themselves belong to linear complexes, has been studied by Sullivan in his thesis,|| and some of the results of the following investigation are direct generalizations of the results obtained by him.

* Second memoir, p. 79.

† *Ibid.*, p. 81.

‡ *Ibid.*, p. 80.

§ *Ibid.*, p. 82.

|| C. T. Sullivan, "Properties of Surfaces Whose Asymptotic Curves Belong to Linear Complexes," *Transactions*, XV, 167-96.

We shall need the fundamental invariants of weights four, nine, and ten of the surfaces R_1 and R_2 . For R_1 , they are

$$(6) \quad \theta = 2^6 \{ a_u'^2 - 2a'[a'_{uu} + 2a'(b_v + f)] \}^* \\ \theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{12} & w_{21} \end{vmatrix}^\dagger$$

$$\theta_{10} = (I^2 - 4J)(K - I'^2) + (II' - 2J')^2 \ddagger$$

We shall denote the corresponding invariants of R_2 by θ' , θ'_9 , θ'_{10} , and they may be obtained from (6) by the interchanges (5).

The computation of θ_9 and θ_{10} will be simplified if R_1 is referred to its flecnodal curves. This is accomplished by the transformation§

$$(7) \quad y = 16a'(\eta + \zeta) \\ z = (8a'_u + \sqrt{\theta})\eta + (8a'_u - \sqrt{\theta})\zeta.$$

The result of this transformation upon (4) is the system

$$(8) \quad \eta_{vv} + \pi_{11}\eta_v + \pi_{11}\zeta_v + \kappa_{11}\eta + \kappa_{12}\zeta = 0, \\ \zeta_{vv} + \pi_{21}\eta_v + \pi_{22}\zeta_v + \kappa_{21}\eta + \kappa_{22}\zeta = 0,$$

where

$$(9) \quad \pi_{11} = \frac{a'_v}{a'} + \frac{\theta_v}{2\theta} + \frac{C}{\sqrt{\theta}}, \quad \pi_{12} = \frac{a'_v}{a'} - \frac{\theta_v}{2\theta} + \frac{C}{\sqrt{\theta}} \\ \pi_{21} = \frac{a'_v}{a'} - \frac{\theta_v}{2\theta} - \frac{C}{\sqrt{\theta}}, \quad \pi_{22} = \frac{a'_v}{a'} + \frac{\theta_v}{2\theta} - \frac{C}{\sqrt{\theta}} \\ C = 8a' \left(\frac{\delta^2 \log a'}{\delta u \delta v} - 4a'b \right).$$

The quantities π'_{ij} , θ' , C' for the equations of R_2 referred to its flecnodal curves may be obtained from (9) by the interchanges (5).

* Second memoir, p. 61.

† *Ibid.*, p. 112.

‡ W, p. 96.

§ Second memoir, p. 83.

2. ANALYTIC CONDITIONS FOR THE OSCULATING RULED SURFACES TO BELONG TO LINEAR COMPLEXES

SPECIAL CASES

The necessary and sufficient conditions for R_1 to belong to a linear complex is that θ_2 vanish identically.* The elements of this determinant are

$$\begin{aligned}
 (10) \quad & u_{12} = u_{21} = 0, \\
 & u_{11} - u_{22} = -\sqrt{\theta}, \quad v_{11} - v_{22} = -2(\sqrt{\theta})_v, \\
 & v_{12} = \sqrt{\theta} \pi_{12}, \quad v_{21} = -\sqrt{\theta} \pi_{21}, \\
 & w_{11} - w_{22} = -4[(\sqrt{\theta})_{vv} + \sqrt{\theta} \pi_{12} \pi_{21}], \\
 & w_{12} = \frac{2}{\sqrt{\theta}} [\theta_v \pi_{12} + \theta (\pi_{12})_v + C \sqrt{\theta} \pi_{12}], \\
 & w_{21} = \frac{2}{\sqrt{\theta}} [-\theta_v \pi_{21} - \theta (\pi_{21})_v + C \sqrt{\theta} \pi_{21}].
 \end{aligned}$$

Introducing these into (6), θ_9 reduces to

$$(11) \quad \theta_2 = 2\theta^{\frac{3}{2}} \begin{vmatrix} \pi_{21} & \pi_{12} \\ (\pi_{21})_v & (\pi_{12})_v + \frac{2C\pi_{12}}{\sqrt{\theta}} \end{vmatrix}.$$

If we compute C_u , making use of the value of θ and the second integrability condition, we shall find

$$(12) \quad 16a'^2 C_u = 2a'_v \theta - a' \theta_v.$$

Thus π_{12} and π_{21} may be written

$$(13) \quad \pi_{12} = \frac{\theta a' C_u + C \sqrt{\theta}}{\theta}, \quad \pi_{21} = \frac{8a' C_u - C \sqrt{\theta}}{\theta}.$$

The substitution of these values into (11) gives

$$(14) \quad \theta_9 = -\frac{4}{\theta} [C(2^6 a'^2 C_u^2 - C^2 \theta) + 8a' C_u C_v \theta + 4a' C C_u \theta_v - 8a'_v C C_u \theta - 8a' C C_{uv} \theta].$$

* W, p. 167.

Again using (11), we obtain

$$(15) \quad \theta_9 = 4[C^3 + 8a'(CC_{uv} - C_u C_v)] = 0$$

as the necessary and sufficient condition for the surface R_1 to belong to a linear complex.

Obviously θ_9 vanishes for $C=0$. This is the case studied by Sullivan. From (11) we see that in this case all the second-order minors of θ_9 also vanish. Therefore R_1 is contained in a linear congruence* and consequently has two straight-line directrices. Hence all of its asymptotic curves belong to linear complexes.† That is: *If $C=0$, the surface R_1 belongs to a linear congruence; moreover, if this is true for every R_1 , the asymptotic curves $u=\text{constant}$ on S all belong to linear complexes.*

Sullivan proved that if the asymptotic curves $u=\text{constant}$ belong to linear complexes, R_1 belongs to a linear congruence. We may prove this geometrically as follows:

The null planes of $P(u_0, v_0)$, on S , in the complexes which osculate $u=u_0$ and $v=v_0$ coincide with the tangent plane to S at $P(u_0, v_0)$. Hence the linear complex which contains $u=u_0$ contains $R_1(u_0)$ also. Let $u=u_0+du$ be onsecutive to $u=u_0$ on S . It intersects $v=v_0$ in a point $P(u_0+du, v_0)$. g_0 , the generator of $R_1(u_0)$ through $P(u_0, v_0)$ passes through $P(u_0+du, v_0)$; moreover, it is the intersection of the osculating planes of $v=v_0$ at $P(u_0, v_0)$ and $P(u_0+du, v_0)$, and accordingly is the intersection of the null planes of $P(u_0, v_0)$ in the complex containing $u=u_0$, and of $P(u_0+du, v_0)$ in the complex containing $u=u_0+du$. This is true for every v . Hence R_1 belongs to the complex which contains $u=u_0$ and to the complex which contains $u=u_0+du$, and therefore belongs to a linear congruence.

There is, of course, a corresponding theorem for the surface R_2 .

The straight-line directrices of R_1 and R_2 are obviously flecnodal tangents.

The invariant θ_9 vanishes also if either π_{12} or π_{21} vanishes. If we compute θ_{10} for R_1 , we shall find

$$(16) \quad \theta_{10} = \theta^2 \pi_{12} \pi_{21}.$$

Thus if $\pi_{12}=0$, R_1 belongs to a special linear complex‡ and consequently has a straight-line directrix, the axis of the special linear complex. It

* *Ibid.*, p. 170.

† *Ibid.*, p. 289.

‡ *Ibid.*, p. 170.

is clear, geometrically, that this line is a flecnodal tangent. In fact, if we make the further transformation*

$$(17) \quad \eta = \bar{\eta} e^{-\frac{1}{2} \int \pi_{11} dv}, \quad \zeta = \bar{\zeta} e^{-\frac{1}{2} \int \pi_{22} dv}$$

on (8), R_1 will still be referred to its flecnodal curves, and as a result we shall have

$$(18) \quad \begin{aligned} \bar{\pi}_{11} &= 0, & \bar{\pi}_{12} &= \pi_{12} e^{-\frac{1}{2} \int \frac{e}{\sqrt{\theta}} dv} \\ \bar{\kappa}_{12} &= \frac{1}{2} (\bar{\pi}_{12})_v, & \bar{\kappa}_{11} &= \frac{\theta_{10}}{16\theta} + \frac{\sqrt{\theta}}{8} + a'_u + g, \end{aligned}$$

which shows that $C_{\bar{\eta}}$ is a straight line if $\pi_{12} = 0$. Similarly, $C_{\bar{\zeta}}$ is a straight line if $\pi_{21} = 0$.

Let us suppose that both families of osculating ruled surfaces, R_1 and R_2 , belong to special linear complexes. At a point $P(u_0, v_0)$ on S , let $l_1(u_0)$ and $l_2(v_0)$ be the axes of the special complexes $C_1(u_0)$ and $C_2(v_0)$ which contain $R_1(u_0)$ and $R_2(v_0)$ respectively. l_1 and l_2 lie on complementary reguli of the osculating quadric Q_0 . They therefore intersect. As u assumes all its values, $l_1(u)$ generates a ruled surface passing through $l_2(v_0)$, and this is true for every v . Hence: *If both families of osculating ruled surfaces belong to special linear complexes, the two sets of axes generate a quadric surface with these sets as complementary reguli.*

If $C \neq 0$, we may write (15) as follows

$$(19) \quad \theta_9 = 4C^2 \left(C + 8a' \frac{\delta^2 \log C}{\delta u \delta v} \right).$$

The invariant of weight nine for R_2 is

$$(20) \quad \theta'_9 = 4C'^2 \left(C' + 8b \frac{\delta^2 \log C'}{\delta u \delta v} \right).$$

Since

$$(21) \quad \begin{aligned} C &= 8a' \left(\frac{\delta^2 \log a'}{\delta u \delta v} - 4a'b \right), \\ C' &= 8b \left(\frac{\delta^2 \log b}{\delta u \delta v} - 4a'b \right), \end{aligned}$$

* W, p. 132.

we find

$$(22) \quad \begin{aligned} 4a'b - \frac{\delta^2 \log a'C'}{\delta u \delta v} &= 0, \\ 4a'b - \frac{\delta^2 \log bC'}{\delta u \delta v} &= 0, \end{aligned}$$

as the conditions, necessary and sufficient, for the two one-parameter families of osculating ruled surfaces of S to belong to linear complexes, but not to linear congruences.

From (22) we obtain by subtraction

$$(23) \quad \frac{\delta^2 \log \frac{a'C}{bC'}}{\delta u \delta v} = 0.$$

Hence, by integration,

$$(24) \quad \frac{a'C}{bC'} = U(u)V(v),$$

where U and V are functions of u alone and v alone respectively. The canonical form (1) and the parametric curves of S are invariant under the transformation*

$$(25) \quad \bar{y} = \kappa \sqrt{\alpha_u \beta_v} y, \quad \bar{u} = \alpha(u), \quad \bar{v} = \beta(v).$$

The results of this transformation are

$$(26) \quad \begin{aligned} \bar{a}' &= \frac{\alpha_u}{\beta_v^2} a', & \bar{b} &= \frac{\beta_v}{\alpha_u^2} b, \\ \bar{C} &= \frac{8}{\beta_v^3} C, & \bar{C}' &= \frac{8}{\alpha_u^3} C'. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\bar{a}' \bar{C}}{\bar{b} \bar{C}'} \left(\frac{\beta_v}{\alpha_u} \right)^6 &= \frac{a' C}{b C'} \\ &= U(u) V(v). \end{aligned}$$

Whence

$$\frac{\bar{a}' \bar{C}}{\bar{b} \bar{C}'} = \alpha_u^6 U \frac{V}{\beta_v}.$$

* First memoir, p. 249.

In particular, the transformation

$$(27) \quad \bar{y} = \sqrt{\frac{d\bar{u}}{du} \frac{d\bar{v}}{dv}} y, \quad \bar{u} = \int \frac{du}{\sqrt{V/U}}, \quad \bar{v} = \int \sqrt{V/U} dv$$

will replace the equations (1) by a system of the same form and for which

$$\phi = a'C - bC' = 0$$

if the osculating ruled surfaces R_1 and R_2 all belong to linear complexes.

3. PROPERTIES OF THE LINEAR COMPLEXES OF THE OSCULATING RULED SURFACES

When the osculating ruled surfaces belong to linear complexes, the linear complexes which contain R_1 and R_2 obviously coincide with C_1 and C_2 which osculate R_1 and R_2 respectively at P_y ; viz.,*

$$C_1 = a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{34}\omega_{34} + a_{42}\omega_{42} = 0, \\ C_2 = b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0,$$

where

$$(28) \quad \begin{aligned} a_{12} &= 0, & a_{13} &= 2^8 a'^2 C, \\ a_{14} &= -8(a'\theta_v - 2a'_v\theta - 16a'a'_u C), \\ a_{23} &= a_{14}, & a_{34} &= -2^9 a'^3 b C, \\ a_{42} &= -\left[C(\theta + 2^6 a'_u{}^2) + 16\theta \frac{a'_u a'_v}{a'} - 8a'_u \theta_v \right], \end{aligned}$$

and the coefficients of C , are obtained from these by the interchanges

$$(29) \quad \begin{pmatrix} a' & u & 2 & \theta & C \\ b & v & 3 & \theta' & C' \end{pmatrix}.$$

The complexes C' and C_2 are in involution.

If we divide C_1 by $2^7 a'$, ($a' \neq 0$, since S is supposed to be non-ruled), and use the relation

$$(12) \quad C_u = \frac{2a'_v\theta - a'\theta_v}{16a'^2}$$

* First memoir, pp. 85, 89.

the coefficients (28) assume the convenient form

$$\begin{aligned}
 (30) \quad & a_{12} = 0, \quad a_{13} = 2a'C, \quad a_{14} = (a'C)_u \\
 & a_{23} = a_{14}, \quad a_{34} = -4a'^2bC \\
 & a_{42} = - \left[\frac{C(\theta + 2^6a_u'^2)}{2^7a'} + a_u' C_u \right].
 \end{aligned}$$

S being non-ruled, C_1 is obviously indeterminate if and only if $C=0$. The corresponding form for C_2 may be obtained from (30) by means of (29).

We shall need the invariants of C_1 and C_2 . They are

$$\begin{aligned}
 (31) \quad & A = \frac{2^6a'^2C_u^2 - C_2\theta}{2^6} = \frac{\theta_{10}}{2^6}, \\
 & B = - \frac{2^6B^2C_v'^2 - C'^2\theta'}{2^6} = - \frac{\theta'_{10}}{2^6}.
 \end{aligned}$$

Let us suppose that the osculating ruled surfaces of the first kind belong to linear complexes. Denote by $R_1(u_0)$ the osculating ruled surface along $u=u_0$ and by $R_1(u_0+du)$ the osculating ruled surface along $u=u_0+du$, and by $C_1(u_0)$, $C_1(u_0+du)$ the linear complexes which contain $R_1(u_0)$ and $R_1(u_0+du)$ respectively. The linear complexes $C_1(u_0)$ and $C_1(u_0+du)$ determine a linear congruence. We proceed to determine the line co-ordinates of its directrices (d'_1 , d''_1).

The increment du displaces the tetrahedron of reference $T \equiv P_\gamma P_z P_\rho P_\sigma$ to $\bar{T} \equiv \bar{P}_\gamma \bar{P}_z \bar{P}_\rho \bar{P}_\sigma$ where

$$\begin{aligned}
 (32) \quad & \bar{y} = y + zdu + \dots, \\
 & \bar{z} = z - (fy + 2b\rho)du + \dots, \\
 & \bar{\rho} = \rho + \sigma du + \dots, \\
 & \bar{\sigma} = \sigma + (ay + 4a'bz - \beta\rho)du + \dots, \\
 & \alpha = 2bg - f_v, \quad \beta = 2b_v + f.
 \end{aligned}$$

If (x_1, x_2, x_3, x_4) are the co-ordinates of a point referred to T and $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ are the co-ordinates of the same point referred to $\bar{T}$, the two expressions

$$x_1y + x_2z + x_3\rho + x_4\sigma, \quad \bar{x}_1\bar{y} + \bar{x}_2\bar{z} + \bar{x}_3\bar{\rho} + \bar{x}_4\bar{\sigma}$$

can differ only by a factor. Consequently, the equations (32) give

$$(33) \quad \begin{aligned} \omega x_1 &= \bar{x}_1 + (-f\bar{x}_2 + a\bar{x}_4)du, \\ \omega x_2 &= \bar{x}_2 + (\bar{x}_1 + 4a'b\bar{x}_4)du, \\ \omega x_3 &= \bar{x}_3 - (2b\bar{x}_2 + \beta\bar{x}_4)du, \\ \omega x_4 &= \bar{x}_4 + \bar{x}_3 du. \end{aligned}$$

whence

$$(34) \quad \begin{aligned} \omega' \bar{x}_1 &= x_1 + (fx_2 - ax_4)du, \\ \omega' \bar{x}_2 &= x_2 - (x_1 + 4a'bx_4)du, \\ \omega' \bar{x}_3 &= x_3 + (2bx_2 + \beta x_4)du, \\ \omega' \bar{x}_4 &= x_4 - x_3 du, \end{aligned}$$

where ω, ω' are proportionality factors. Hence, if ω_{ij} and $\bar{\omega}_{ij}$ denote the line co-ordinates of a line referred to $\bar{T}$ and T respectively, we find

$$(35) \quad \begin{aligned} \bar{\omega}_{12} &= \omega_{12} - (4a'b\omega_{14} + a\omega_{42})du, \\ \bar{\omega}_{13} &= \omega_{13} + (2b\omega_{12} + \beta\omega_{14} + f\omega_{23} + a\omega_{34})du, \\ \bar{\omega}_{14} &= \omega_{14} - (\omega_{13} + f\omega_{42})du, \\ \bar{\omega}_{23} &= \omega_{23} - (\omega_{13} - 4a'b\omega_{34} + \beta\omega_{42})du, \\ \bar{\omega}_{34} &= \omega_{34} - 2b\omega_{42}du, \\ \bar{\omega}_{42} &= \omega_{42} + (\omega_{14} + \omega_{23})du. \end{aligned}$$

The complex $C_1(u_0 + du)$, referred to T , has for its equation

$$(36) \quad [a_{12} + (a_{12})_u du + \dots] \bar{\omega}_{12} \\ + \dots + [a_{42} + (a_{42})_u du + \dots] \omega_{42} = 0.$$

Making use of (35), and neglecting powers of du higher than the first, we get

$$(37) \quad C_1(u_0 + du) = C_1(u_0) + du \bar{C}_1(u_0 + du)$$

where the coefficients of $\bar{C}_1$ are

$$(38) \quad \begin{aligned} \bar{a}_{12} &= 2ba_{13}, \\ \bar{a}_{13} &= (a_{13})_u - 2a_{23}, \\ \bar{a}_{14} &= (a_{14})_u + 2\beta a_{13} + a_{42}, \\ \bar{a}_{23} &= (a_{23})_u - fa_{13} + a_{42}, \\ \bar{a}_{34} &= (a_{34})_u + aa_{13} + 4a'b a_{23}, \\ \bar{a}_{42} &= (a_{42})_u - 2(b_v + f)a_{23} - 2ba_{34}. \end{aligned}$$

Substituting the values of the coefficients of C_1 from (30) into (38) we get

$$\begin{aligned}
 \bar{a}_{12} &= 4a'bC, \\
 \bar{a}_{13} &= 0, \\
 \bar{a}_{14} &= E + 2a'b_vC, \\
 \bar{a}_{23} &= E - 2a'b_vC, \\
 \bar{a}_{34} &= \frac{a'C}{2^6b}(2^6b_v^2 - \theta'), \\
 \bar{a}_{4i} &= -\frac{a'_u}{a}E + \frac{C}{2^7a'}(2^{10}a'^3b^2 - \theta_u),
 \end{aligned}
 \tag{39}$$

where

$$E = \frac{2^6a'(a'C_u)_u - C\theta}{2^6a'}.$$

The directrices (d'_1 , d''_1) coincide with the axes of the two special complexes of the pencil

$$\lambda C_1 + \mu \bar{C} = 0.$$

The special complexes of this pencil are given by the values of $\omega = \frac{\lambda}{\mu}$ which satisfy the quadratic equation

$$A\omega^2 + (A, \bar{A})\omega + \bar{A} = 0,$$

where A and $\bar{A}$, the invariants of C_1 and $\bar{C}_1$, respectively, and $(A, \bar{A})$, the bilinear invariant of C_1 and $\bar{C}_1$, have the values

$$\begin{aligned}
 A &= \frac{\theta_{10}}{2^6}, \quad (A, \bar{A}) = \frac{(\theta_{10})_u}{2^6}, \\
 \bar{A} &= E^2 - \frac{a'^2C^2\theta'}{16}.
 \end{aligned}
 \tag{42}$$

Denoting the roots of (41) by ω_i , $i = 1, 2$, and substituting in (40), we get

$$\begin{aligned}
 \omega_{12} &= \frac{a'C}{2^6b}(2^6b_v^2 - \theta') - 4a'^2bC\omega_i \\
 \omega_{13} &= -\frac{a'_u}{a'}E + \frac{C}{2^7a'}(2^{10}a'^3b - \theta_u) - [C(\theta + 2^6a'^2) + 2^7a'a'_uC_u] \frac{\omega_i}{2^7a'}, \\
 \omega_{14} &= E - 2a'b_vC + (a'C)_u\omega_i, \\
 \omega_{23} &= E + 2a'b_vC + (a'C)_u\omega_i, \\
 \omega_{34} &= 4a'bC, \\
 \omega_{42} &= 2a'C\omega_i,
 \end{aligned}
 \tag{43}$$

as the line co-ordinates of d'_1 and d''_1 .

The line co-ordinates of the directrices (d'_2, d''_2) of the linear congruence common to the linear complexes $C_2(v_0)$ and $C_2(v_0 + dv)$ may be obtained from those of (43) by performing (29) and replacing ω_i by ω'_i , where ω'_i are the two roots of the equation

$$(44) \quad B\omega'^2 + (B, \bar{B})\omega' + \bar{B} = 0,$$

B and $\bar{B}$ being the invariants of C_2 and $\bar{C}_2$ respectively.

We find by direct substitution that d'_1 and d''_1 belong to $C_2(v_0)$ and to

$$(45) \quad C' = b_v\omega_{34} + b(\omega_{14} - \omega_{23}) = 0,$$

the linear complex which osculates the curve $v = v_0$ at $P(u_0, v_0)$. Similarly, it can be shown that the lines d'_2 and d''_2 belong to $C_1(u_0)$ and to

$$(46) \quad C'' = -a'_u\omega_{42} + a'(\omega_{14} + \omega_{23}) = 0$$

which osculates $u = u_0$ at $P(u_0, v_0)$. By making use of the identity

$$a'\theta'_v - b\theta_u = 0,$$

which may be verified by performing the indicated differentiations and using (2), it can be shown that the co-ordinates of the lines d'_1 and d''_1 satisfy the conditions that they intersect both the lines d'_2 and d''_2 . That is, the four directrices $(d'_1, d''_1, d'_2, d''_2)$ form a skew quadrilateral. However, it is useful to establish this fact geometrically.

Let u be fixed and v vary. Then, if the surfaces $R_1(u)$ belong to linear complexes, $C_1(u_0)$ and $C_1(u_0 + du)$ remain fixed. Therefore d'_1 and d''_1 remain fixed, and hence belong to all the linear complexes $C_2(v)$. In particular, they belong to $C_2(v_0)$ and $C_2(v_0 + dv)$ and accordingly intersect d'_2 and d''_2 . Moreover, if the surfaces $R_2(v)$ belong to linear complexes $C_2(v)$, and v remains fixed while u assumes all its values, the lines d'_2 and d''_2 remain fixed and therefore intersect all the lines d'_1 and d''_1 , and this is true for every value of v . Hence, all of the lines d'_1, d''_1 intersect all of the lines d'_2, d''_2 . We therefore have the theorem

If each family of osculating ruled surfaces of a surface belongs to a family of linear complexes, the directrices of the congruences determined by consecutive complexes of one family generate a regulus of a quadric and the directrices of the congruences determined by consecutive complexes of the other family generate the complementary regulus of the same quadric.

We shall denote this quadric by Q .

Consider the rulings of the second kind on the osculating quadric Q_0 . An arbitrary ruling is given by the line joining the points*

$$\tau_1 = ay + \beta z, \quad \tau_2 = -4a'b\beta_y - 2a\rho + 2\beta\sigma.$$

Therefore, the line co-ordinates of an arbitrary generator are

$$(47) \quad \begin{aligned} \omega_{12} &= 2a'b\beta^2, & \omega_{23} &= a\beta, \\ \omega_{13} &= a^2, & \omega_{34} &= 0, \\ \omega_{14} &= a\beta, & \omega_{42} &= -\beta^2. \end{aligned}$$

If d'_1 and d''_1 are on Q_0 , their co-ordinates must be proportional to these for some values of a and β , since the rulings of the second kind on Q_0 do not belong to $C_2(v_0)$ and $C'(v_0)$. Then the co-ordinate $\omega_{34} = 4a'bC$ of d'_1 and d''_1 must vanish if these directrices lie on Q_0 . This can happen only if $C=0$, since S is assumed to be non-ruled. Therefore, if $R_1(u_0)$ belongs to a linear complex, but not to a linear congruence, the lines d'_1 and d''_1 do not lie on $Q_0(u_0, v_0)$.

There is a corresponding theorem for $R_2(v_0)$.

If R_1 belongs to a linear congruence (that is, if $C=0$), the complex C_1 is indeterminate. One complex of the pencil, determined by the congruence, which contains $R_1(u_0)$ is $C''(u_0)$ as we have already seen. We proved also that $C''(u_0+du)$ contains $R_1(u_0)$ as well as $R_1(u_0+du)$. The congruence determined by $C''(u_0)$ and $C''(u_0+du)$ therefore coincides with the congruence containing $R_1(u_0)$ and consequently its directrices are the flecnodal tangents to $R_1(u_0)$. In an analogous manner it can be shown that the directrices of the congruence common to $C'(v_0)$ and $C'(v_0+dv)$ are the flecnodal tangents to $R_2(v_0)$, and the proof that the loci of these sets of flecnodal tangents are complementary reguli on a quadric is the same as for the directrices (d'_1, d''_1) , (d'_2, d''_2) . Sullivan called the quadric surface in this case a *directrix quadric*.

The directrices (d'_1, d''_1) intersect $Q_0(u_0, v_0)$ in points which lie on the flecnodal tangents to $R_2(v_0)$. For, consider three consecutive generators g_{-1} , g_0 , g_1 of $R_2(v_0)$. They determine the quadric $Q_0(u_0, v_0)$ osculating $R_2(v_0)$ along g_0 . Consider also a fourth generator g_2 consecutive to g_1 . The lines g_0 , g_1 , g_2 determine the quadric $Q_0(u_0+du, v_0)$ osculating $R_2(v_0)$ along g_1 . The flecnodal tangents to $R_2(v_0)$ on g_0 intersect g_{-1} , g_0 , g_1 , g_2 and accordingly are rulings of the first kind on both $Q_0(u_0, v_0)$ and $Q_0(u_0+du, v_0)$. But $C_1(u_0)$ contains three, and therefore all, rulings of the first kind of $Q_0(u_0, v_0)$ and $C_1(u_0+du)$ contains the rulings of the first kind on $Q_0(u_0+du, v_0)$. Therefore, the flecnodal tangents to $R_2(v_0)$ on g_0 belong to both $C_1(u_0)$ and $C_1(u_0+du)$ and hence intersect d'_1 and d''_1 . In a similar

* W, p. 146.

manner it can be shown that the flecnode tangents to $R_2(v_0)$ intersect the lines d'_2 and d''_2 .

Let us suppose, if possible, that all the complexes $C_1(u)$ are contained in a pencil. Then all the congruences determined by pairs of complexes $C_1(u)$, $C_1(u+du)$ would have the same directrices (d'_1 d''_1) for every value of u . But all the flecnode tangents to $R_2(v)$ would then have two straight-line intersectors. In particular, the flecnode tangents to $R_2(v_0)$ on two consecutive generators g_{-1} , g_0 all on $Q_0(u_0, v_0)$ would intersect d'_1 and d''_1 . Hence d'_1 and d''_1 would intersect four, and therefore all, rulings of the first kind on $Q_0(u_0, v_0)$. But we proved that d'_1 and d''_1 cannot lie on $Q_0(u_0, v_0)$ unless either R_1 belongs to a linear congruence or S is ruled. Similarly, the complexes $C_2(v)$ do not belong to a pencil. Therefore:

If the osculating ruled surfaces of a non-ruled surface S belong to linear complexes, but not to linear congruences, neither family of these complexes can belong to a pencil.

If all the linear complexes $C_1(u)$ belong to a one-parameter family of linear complexes not contained in a net, the family could have at most two lines in common. But we proved that the lines d'_2 and d''_2 belong to all the complexes $C_1(u)$. Therefore if these complexes belong to a hypernet, all the pairs of lines (d'_2 , d''_2) coincide, and the family of complexes $C_2(v)$ belong to a pencil. But by the theorem just proved S is either ruled or R_2 belongs to a linear congruence. In a similar manner it can be shown that the linear complexes $C_2(v)$ belong to a net. Hence the theorem:

If both families of osculating ruled surfaces of a non-ruled surface S belong to linear complexes, but not to linear congruences, these linear complexes must form two one-parameter families belonging to two involutory nets.

Consider now the congruence common to the linear complexes C_2 and C' . Denote its directrices by l'_2 and l''_2 , which may be found by determining the axes of the two special complexes of the pencil

$$(48) \quad C_2 + \lambda C' = 0.$$

The line co-ordinates of l'_1 and l''_1 determined in the usual manner are

$$(49) \quad \begin{aligned} \omega_{12} &= \frac{C'(\theta' + 2^6 b_v^2) + 2^7 b b_v C'}{2^7 b} - b_v \lambda, \\ \omega_{13} &= 4a' b^2 C', \\ \omega_{14} &= -(bC')_v + b\lambda, \\ \omega_{13} &= -\omega_{14}, \\ \omega_{34} &= 2bC', \\ \omega_{42} &= 0, \end{aligned}$$

where

$$\lambda = 16(8bC'_v \pm C'v'\theta').$$

The line co-ordinates of an arbitrary generator of the first kind on Q_0 are found to be

$$(50) \quad \begin{aligned} \omega_{12} &= \alpha^2, \\ \omega_{13} &= 2\alpha'b\beta', \\ \omega_{14} &= \alpha\beta', \\ \omega_{23} &= -\alpha\beta, \\ \omega_{34} &= \beta^2, \\ \omega_{42} &= 0, \end{aligned}$$

where α, β are arbitrary functions of u and v . The values of α and β for which (50) are proportional to (49) can be readily determined. Therefore the directrices (l'_2, l''_2) lie on Q_0 . But d'_1 and d''_1 belong to C_2 and C' and hence intersect l'_2 and l''_2 . They also intersect the flecnodal tangents to R_2 , and since they do not lie on Q_0 , it follows that *the directrices of the congruence determined by C_2 and C' coincide with the flecnodal tangents to R_2* . It is not difficult to show this in general by determining the line co-ordinates of the flecnodal tangents. There is of course a similar property of the congruence common to C'_1 and C'' whose directrices we denote by (l'_1, l''_1).

We see that l'_1 and l''_1 belong to $C_2(v)$ and $C'(v)$, since these complexes contain all the generators of the second kind on Q_0 ; and since they intersect d'_2 and d''_2 , they belong also to $C_2(v+dv)$. We have already seen that d'_1 and d''_1 belong to these three complexes. The lines l'_2, l''_2, d'_2, d''_2 belong to the complexes $C_1(u), C_1(u+du), C''(u)$. Therefore, *the lines l'_1, l''_1, d'_1, d''_1 are rulings of the regulus common to the three linear complexes $C'(v), C_2(v), C_2(v+dv)$, and the lines l'_2, l''_2, d'_2, d''_2 are rulings of the complementary regulus which is common to the three linear complexes $C'(u), C_1(u), C_1(u+du)$* .

4. A GEOMETRIC CONSTRUCTION FOR THE SURFACES OF THE PROBLEM

We now proceed to give a geometric construction for surfaces whose two families of osculating ruled surfaces belong respectively to two families of linear complexes contained in a net.

Select arbitrarily a quadric Q and a one-parameter family of pairs of lines on each regulus of Q . Each pair determines a pencil of linear complexes, the pair of lines being the axes of the special complexes of the

pencil. According to some arbitrary law, pick out one complex from each of these pencils, giving two one-parameter families of linear complexes contained in nets. Denote by $[d'_1(u_0), d''_1(u_0)]$, $[d'_1(u_0+du), d''_1(u_0+du)]$, $[d'_1(u_0+2du), d''_1(u_0+2du)]$, $\dots$ the pairs of lines corresponding to $u=u_0$, $u=u_0+du$, $u=u_0+2du$, $\dots$, and by $C_1(u_0)$, $C_1(u_0+du)$, $C_1(u_0+2du)$, $\dots$ the corresponding linear complexes (selected according to an arbitrary law). Denote by $[d'_2(v_0), d''_2(v_0)]$, $[d'_2(v_0+dv), d''_2(v_0+dv)]$, $[d'_2(v_0+2dv), d''_2(v_0+2dv)]$, $\dots$ and $C_2(v_0)$, $C_2(v_0+dv)$, $C_2(v_0+2dv)$, $\dots$ the pairs of lines and complexes corresponding to $v=v_0$, $v=v_0+dv$, $v=v_0+2dv$, $\dots$. Let $S_1(u_0)$ be an arbitrary ruled surface through the lines $d'_1(u_0)$ and $d''_1(u_0)$. $S_1(u_0)$ clearly belongs to $C_1(u_0)$. Let $h_0(u_0)$, $h_1(u_0)$, $h_2(u_0)$, $\dots$ be the generators of $S_1(u_0)$ corresponding to $v=v_0$, $v=v_0+dv$, $v=v_0+2dv$, $\dots$. Pick out an arbitrary point P_0 on $h_0(u_0)$. The null plane of P_0 in $C_2(v_0)$ intersects $h_1(u_0)$ in a point P_1 . Let $g'_0(v_0)$ be the line through P_0 and P_1 . It belongs to $C_2(v_0)$. The null plane of P_1 in $C_2(v_0+dv)$ intersects $h_2(u_0)$ in a point P_2 . Let $g'_1(v_0+dv)$ be the line joining P_1 to P_2 . It belongs to $C_2(v_0+dv)$. Proceeding in this manner, we obtain in the limit as dv approaches zero a curve $P_0P_1P_2\dots$ on $S_1(u_0)$ which we shall call $\Gamma(u_0)$. The plane determined by the points P_{k-1} , P_k , P_{k+1} becomes, in the limit, the osculating plane of $\Gamma(u_0)$ at the point P_k . The null plane of P_k in $C_1(u_0)$ intersects the plane $P_{k-1}P_kP_{k+1}$ in a line $g_k(u_0)$. The ruled surface $R_1(u_0)$ determined by the lines g_k , $\kappa=0, 1, 2, \dots$ belongs to the linear complex $C_1(u_0)$, and clearly has, in the limit, the curve $\Gamma(u_0)$ as an asymptotic curve on it. Next, let P'_0 be a point on $g_0(u_0)$ ultimately to be allowed to approach the corresponding point P_0 as its limit. The null plane of P'_0 in the complex $C_2(v_0)$ intersects $g_1(u_0)$ in a point P'_1 . The line through P'_0 , P_1 , say $g'_1(v_0)$, belongs to $C_2(v_0)$. The null plane of P'_1 in the complex $C_2(v_0+dv)$ intersects $g_2(u_0)$ in P'_2 . The line $g'_1(v_0+dv)$ through P'_1 , P'_2 belongs to $C_2(v_0+dv)$. As we continue this process we obtain points P'_κ on $g_\kappa(u_0)$ such that $g'_\kappa(v_0+\kappa dv)$ through P'_κ , $P'_{\kappa+1}$ belongs to the complex $C_2(v_0+\kappa dv)$, $\kappa=0, 1, 2, \dots$. For later use, draw the lines $h_\kappa(u_0+du)$ from P'_κ to the pair of lines $d'_1(u_0+du)$, $d''_1(u_0+du)$ obtaining, for $\kappa=0, 1, 2, \dots$, a ruled surface $S_1(u_0+du)$ belonging to $C_1(u_0+du)$. Now at P'_κ draw the line $g_\kappa(u_0+du)$ which lies both in the plane $P'_{\kappa-1}P'_\kappa P'_{\kappa+1}$ and in the null plane of P'_κ in $C_1(u_0+du)$. The lines $g_\kappa(u_0+du)$, $\kappa=0, 1, 2, \dots$, determine a ruled surface $R_1(u_0+du)$ which belongs to $C_1(u_0+du)$, and has in the limit (as $dv \rightarrow 0$) the curve $\Gamma(u_0+du)=P'_0P'_1P'_2\dots$ as an asymptotic curve on it. By continuing this process we obtain $R_1(u_0+ndu)$ and $S_1(u_0+ndu)$, belonging

to $C_1(u_0+ndu)$ and $\Gamma(u_0+ndu)=P_0^{(n)}P_1^{(n)}P_2^{(n)}\dots$ as an asymptotic curve on $R_1(u_0+ndu)$, $n=0, 1, 2, \dots$. Furthermore, the ruled surface $R_2(v_0)$ determined by the lines $g'_\kappa(v_0)$, $\kappa=0, 1, 2, \dots$, belongs to $C_2(v_0)$, and has in the limit, as du approaches zero, $\Gamma'(v_0)=P_0P'_0P''_0\dots$ as an asymptotic curve on it. Similarly, the ruled surface $R_2(v_0+ndv)$ determined by $g'_\kappa(v_0+ndv)$, $\kappa=0, 1, 2, \dots$, belongs to $C_2(v_0+ndv)$ and has in the limit the curve $\Gamma'(v_0+ndv)=P_nP'_nP''_n\dots$ as an asymptotic curve on it. Finally we pass to the limit. The surfaces $R_1(u_0+ndu)$, $n=0, 1, 2, \dots$, just constructed have as an envelope (as $du \rightarrow 0$) a curved surface S with the one family of osculating ruled surfaces contained in one of the given one-parameter families of linear complexes; and the surfaces $R_2(v_0+ndv)$, $n=0, 1, 2, \dots$, have as an envelope (as $dv \rightarrow 0$) the same surface S with the second family of osculating ruled surfaces contained in the other given one-parameter family of linear complexes.

If it should happen that the curves $\Gamma(u_0+ndu)$ are asymptotic on the surfaces $S_1(u_0+ndu)$, the surfaces $R_1(u_0+ndu)$ will then coincide with the surfaces $S_1(u_0+ndu)$, and will therefore belong to linear congruences. But this is in accordance with our theory; for all the asymptotic curves on $S_1(u_0+ndu)$ belong to linear complexes, so that the envelope of these surfaces would then have one family of its asymptotic curves contained in linear complexes, and this implies that the corresponding family of osculating ruled surfaces belong to linear congruences as we proved.

